

Scattering of long waves by cylindrical obstacles and gratings using matched asymptotic expansions

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The method of matched asymptotic expansions is applied to several long-wave problems including the scattering of acoustic waves by a grating of cylinders and the scattering of water waves incident on horizontal cylinders. It is shown that a naïve application of the method can lead to incorrect results. A modified expansion procedure is developed and applied to a number of problems.

1. Introduction

The method of matched asymptotic expansions has been used to treat a number of interesting problems, involving interaction of waves and obstacles; see e.g. Tuck (1975). In the simplest problems, there are two lengthscales, namely a wavelength λ and a typical diameter of the obstacles d ; here we assume that $d \ll \lambda$. Such problems are well understood. Suppose now that the obstacles have two lengthscales. A prototypical example is a grating, composed of an infinite row of identical equally spaced cylinders. Lamb (1898, 1932, §307) considered the scattering of long acoustic waves by a grating of small circular cylinders; more precisely, with wavenumber $k = 2\pi/\lambda$, spacing $2h$ and diameter $2a$, he assumed that $kh \ll 1$ and $a/h \ll 1$. He used a method similar to that of matched asymptotic expansions and obtained a simple formula for the reflection coefficient (see §2.1 below). Later he used his method to derive a formula for the wavelength of the fundamental (antisymmetric) standing wave in a long rectangular tank containing a small vertical cylinder at its centre, and verified his result experimentally (Lamb & Cook 1910).

Lamb's method is simple and has been used widely. Moreover, it is not limited to small circular cylinders; the essential assumption is that $kh \ll 1$. However, it has been known for thirty years that Lamb's formula for the reflection coefficient for waves at normal incidence to a grating of small circular cylinders is incorrect (Twersky 1956, 1962). In the present paper, we show how to modify Lamb's method so as to obtain the correct result.

The experiment of Lamb & Cook exploits the analogy between two-dimensional acoustics and three-dimensional water waves. Thus, for water-wave scattering by a row of vertical cylinders, we can separate out the dependence on depth, leaving a grating problem in the horizontal plane. This water-wave problem has been studied for several reasons, for example to elucidate the behaviour of waves near a pile-supported offshore structure or to examine the use of a row of piles for a breakwater, with the reflection of the waves being an important phenomenon. For circular

cylinders, Twersky (1956, 1962), Spring & Monkmeyer (1975), Miles (1983) and others provide means for calculating the wave field for waves in any depth. Here we examine long waves incident on a pile array, neglecting dissipation by turbulence and flow separation around the piles (see Hayashi, Kano & Shirai 1966, for the case of narrowly spaced piles).

The method of matched asymptotic expansions has also been used to treat the interaction of shallow-water waves with horizontal cylinders. Guiney, Noye & Tuck (1972) consider scattering by a fixed cylinder in the free surface. They also examine a submerged ridge, as does Tuck (1977), whereas Miles (1982*b*) considers a trench. All of these authors' results can be put into the form of Lamb's formula. We again modify Lamb's method and show that his result is correct for submerged ridges and trenches, but not for surface-piercing cylinders.

In the next section we discuss the case of acoustic waves incident on a grating. This problem is equivalent to water-wave scattering by a row of vertical cylinders (or to a single cylinder on the centreline of a wave tank). Lamb's solution is reviewed and the modified matched asymptotic expansion is presented, giving new approximations for the reflection and transmission coefficients. These approximations are compared to Twersky's exact solution and with Miles's (1982*a*) solution for small cylinders.

In §4, we consider the scattering of shallow-water waves by horizontal cylinders. Again, we obtain new approximations for the reflection and transmission coefficients and compare them with known solutions.

An important parameter in our approximations is the blockage coefficient l for the potential flow past an obstacle in a channel. Several new results for l are obtained in our analyses. For large circular cylinders ($a/h \leq 0.5$), we use a conformal mapping due to Richmond (1923) and, for small cylinders, we obtain l in terms of the added mass of a single cylinder translating in an unbounded stationary fluid.

2. Gratings and waveguides

We shall first treat the diffraction of acoustic waves by an infinite row of equally spaced identical cylinders immersed in a compressible fluid. The cylinders will be assumed to be symmetric about two lines, one of which is the y -axis. An example of such a grating of cylinders would be a line of circular cylinders with their centres at $y = \pm 2n\hbar$ for $n = 0, 1, 2, \dots$, or elliptical cylinders with their major (or minor) axes along the y -axis.

A plane sound wave is normally incident upon the grating of cylinders; its velocity potential is $\text{Re} \{ \phi^{\text{in}}(x, y) e^{-i\omega t} \}$, where Re denotes the real part, with

$$\phi^{\text{in}}(x, y) = e^{ikx}. \quad (1)$$

(Henceforth we shall suppress the time dependence.) The incident wave is propagating in the positive x -direction, with wavelength $2\pi/k$ and radian frequency ω . The governing equation for the (total) velocity potential $\phi(x, y)$ within the fluid surrounding the cylinders is the Helmholtz equation:

$$(\nabla^2 + k^2) \phi \equiv \phi_{xx} + \phi_{yy} + k^2 \phi = 0. \quad (2)$$

On each cylinder within the grating there is a no-flow condition

$$(\phi)_n = 0, \quad (3)$$

where the subscript n denotes a normal derivative, directed into the fluid. In addition, the scattered potential

$$\phi^{sc} \equiv \phi - \phi^{in} \tag{4}$$

must satisfy a radiation condition as $|x| \rightarrow \infty$.

This acoustics problem for $\phi(x, y)$ is directly analogous to the problem of water-wave scattering by a row of identical vertical cylinders, piercing the horizontal bottom and the free surface. (The case of horizontal cylinders in shallow water will be discussed in §4.)

The radiation condition can be stated explicitly in terms of both ϕ and ϕ^{sc} :

$$\phi \sim \begin{cases} T e^{ikx} & \text{as } x \rightarrow \infty, \\ e^{ikx} + R e^{-ikx} & \text{as } x \rightarrow -\infty \end{cases} \tag{5}$$

or

$$\phi^{sc} \sim \begin{cases} (T-1) e^{ikx} & \text{as } x \rightarrow \infty, \\ R e^{-ikx} & \text{as } x \rightarrow -\infty. \end{cases} \tag{6}$$

Here, R and T are the (complex) reflection and transmission coefficients, respectively. Energy and reciprocity considerations imply that R and T must satisfy (see e.g. Twersky 1962, equation 38)

$$|R|^2 + |T|^2 = 1 \tag{7}$$

and

$$RT^* + R^*T = 0, \tag{8}$$

where the asterisk denotes the complex conjugate, and the symmetry of the cylinders about the y -axis has been used to obtain the last equation. These equations are equivalent to

$$|R+T| = 1 \tag{9}$$

and

$$|R-T| = 1. \tag{10}$$

Utilizing the symmetry of this problem, we can reduce it to a *waveguide* problem, with two rigid parallel walls, one at $y = 0$ and the other at $y = h$. A symmetric protrusion Γ of width $2b$, is situated on one of these walls, say $y = 0$; the finite region bounded by Γ and $y = 0$ is denoted by D_- . (Note that Γ can be a depression, i.e. Γ can be in the region $y < 0$.) This waveguide problem is equivalent to the scattering of the ‘dominant’ TE (or H_{10}) mode by a capacitive post in a rectangular waveguide; for small circular posts, see e.g. Lewin (1951, §2.3; 1975, §5.3) and Marcuvitz (1951, §5.13).

2.1. Lamb’s method

Lamb (1898, 1932, §307) has described a method for solving these problems for the case of small circular cylinders. He begins by assuming that ϕ can be expressed as

$$\phi = T e^{ikx} + \sum_{m=1}^{\infty} D_m e^{-\lambda_m x} \cos\left(\frac{m\pi y}{h}\right) \quad \text{for } x > 0 \tag{11}$$

and

$$\phi = e^{ikx} + R e^{-ikx} - \sum_{m=1}^{\infty} D_m e^{\lambda_m x} \cos\left(\frac{m\pi y}{h}\right) \quad \text{for } x < 0. \tag{12}$$

Here, D_m are constants and

$$\lambda_m = \left\{ \frac{m^2 \pi^2}{h^2} - k^2 \right\}^{\frac{1}{2}} \approx \frac{m\pi}{h},$$

since kh was assumed to be small. If $k|x|$ is also small, we obtain

$$\phi \approx T + ikxT + \sum_{m=1}^{\infty} D_m e^{-m\pi x/h} \cos\left(\frac{m\pi y}{h}\right) \quad \text{for } x > 0, \tag{13}$$

$$\phi \approx 1 + R + ikx(1 - R) - \sum_{m=1}^{\infty} D_m e^{m\pi x/h} \cos\left(\frac{m\pi y}{h}\right) \quad \text{for } x < 0. \tag{14}$$

Lamb then examined a related problem for potential flow, using the same geometry :

$$\nabla^2 \psi_a = 0 \text{ in the fluid,} \tag{15}$$

$$(\psi_a)_n = 0 \text{ on the boundaries,} \tag{16}$$

$$\psi_a \sim \frac{x}{h} \text{ as } |x| \rightarrow \infty \tag{17}$$

This corresponds to a uniform flow past circular cylinders, with unit flux at $|x| = \infty$. For *small* cylinders, the solution to this problem (to within an arbitrary additive constant) is given approximately by Lamb (1932, §64) as

$$\psi_a = \frac{x}{h} + C \frac{\sinh(\pi x/h)}{\cosh(\pi x/h) - \cos(\pi y/h)} \tag{18}$$

$$= \frac{x}{h} + C \operatorname{sgn}\left(\frac{x}{h}\right) \sum_{m=0}^{\infty} \epsilon_m e^{-m\pi|x/h|} \cos\left(\frac{m\pi y}{h}\right), \tag{19}$$

where $\epsilon_0 = 1$, $\epsilon_m = 2$ for $m > 0$, and

$$C = \frac{1}{2}\pi \left(\frac{a}{h}\right)^2. \tag{20}$$

Comparing this potential to (13) and (14), we see, for small $k|x|$, that

$$\phi = A\psi_a + B, \tag{21}$$

provided that we choose A and B such that

$$T = AC + B, \quad ikT = \frac{A}{h}, \quad 1 + R = -AC + B, \quad ik(1 - R) = \frac{A}{h}, \tag{22}$$

and $D_m = 2AC$ for $m = 1, 2, \dots$. Solving these equations yields $AC = -R$, $B = 1$ and

$$R = -\frac{ikl}{1 - ikl}, \quad T = \frac{1}{1 - ikl}, \tag{23}$$

where we have defined l by $l = Ch$. (24)

These formulae for R and T satisfy the energy conditions (7)–(10) identically; in fact they satisfy

$$R + T = 1. \tag{25}$$

Lamb’s method is a form of matched asymptotic expansion. The assumed solutions (11), (12) are the outer solutions which satisfy the Helmholtz equation (2), the radiation condition, and the boundary conditions on the flat walls, but do not satisfy the boundary condition on the protrusion Γ . On the other hand, ψ_a is an inner solution that satisfies all the boundary conditions and Laplace’s equation, which is a reasonable approximation to the Helmholtz equation for small kh and for $x = O(h)$,

$y = O(h)$. Then we see that Lamb matches these two solutions at $x = \pm X$, where $kX \ll kh \ll 1$ (or $kX \rightarrow 0$ and $X/h \rightarrow 0$).

If we match further away from the cylinder, so that $kh \ll kX \ll 1$ (or $kX \rightarrow 0$, but $X/h \rightarrow \infty$), then the evanescent terms in (11) and (12) can be discarded. Thus, we have

$$\phi = \begin{cases} T e^{ikx} & \text{for } x > 0, \\ e^{ikx} + R e^{-ikx} & \text{for } x < 0. \end{cases} \tag{26}$$

For small $k|x|$, these give

$$\phi \sim \begin{cases} T + ikxT, & x > 0, \\ 1 + R + ikx(1 - R), & x < 0. \end{cases} \tag{27}$$

$$\tag{28}$$

This is the inner expansion of the outer solution. The inner solution is supposed to be

$$\Phi = A\psi_a + B, \tag{29}$$

as in (21). Its outer expansion is

$$\Phi \sim (x + l \operatorname{sgn} x) \frac{A}{h} + B \quad \text{as } \frac{|x|}{h} \rightarrow \infty, \tag{30}$$

since

$$h\psi_a \sim x + l \operatorname{sgn} x \quad \text{as } |x| \rightarrow \infty. \tag{31}$$

We see that (30) and (28) agree if R, T, A and B satisfy the same equations as before, namely (22) and (24), leading to the same formulae, (23), for R and T .

The result (31) holds for uniform flow past any protrusion or depression Γ . The parameter l is called the *blockage coefficient*; it depends on the shape of Γ and the width of the strip h , and it can be computed numerically or analytically (for a more complete discussion, see Tuck 1975, §V). We note that with this modified matching procedure, we do not need to know ψ_a explicitly in order to find R and T ; it is sufficient to know the blockage coefficient, as defined by (31). Moreover, we do not assume that Γ is small; it is sufficient that the diameter of Γ be $O(h)$.

For small kl , (23) gives

$$R \sim -ikl, \quad T \sim 1 + ikl. \tag{32}$$

In particular, for small circular cylinders, we can use (20) and (24) to give

$$R \sim -\frac{i\pi ka^2}{2h}, \quad T \sim 1 + \frac{i\pi ka^2}{2h}. \tag{33}$$

Kakuno (1983) applied Lamb's method to the scattering of water waves by a grating. His results, for circular and rectangular cylinders, can be written in Lamb's form, (23).

Using either of the two matching procedures above, however, gives *incorrect* results! For circular cylinders, the exact solution has been given by Twersky (1956, 1962). (Burke & Twersky 1966 have also solved the problem of plane waves obliquely incident on a grating of elliptical cylinders.) His results (1962, eq. 120) are

$$R \sim -\frac{3i\pi ka^2}{4h}, \quad T \sim 1 + \frac{i\pi ka^2}{4h}. \tag{34}$$

These approximations were also obtained, using different methods, by Marcuvitz (1951) and Lewin (1951, 1975); a somewhat simplified version of Lewin's 'multiplet theory' is described in Appendix A.

Exact solutions for ϕ are not available for arbitrary geometries. Therefore it is worthwhile to develop a correct matching procedure.

2.2 Modified matching procedure

We begin by separating the inner problem into two simpler problems, one symmetric about $x = 0$, and one antisymmetric. Thus we can write

$$\phi = \phi_a + \phi_s, \quad \phi^{sc} = \phi_a^{sc} + \phi_s^{sc}, \quad \phi^{in} = \phi_a^{in} + \phi_s^{in}, \quad (35)$$

where e.g.
$$\phi_a = \frac{1}{2}[\phi(x, y) - \phi(-x, y)] \quad (36)$$

and
$$\phi_s = \frac{1}{2}[\phi(x, y) + \phi(-x, y)] \quad (37)$$

are antisymmetric and symmetric, respectively, about $x = 0$. From (1), we have

$$\phi_s^{in} = \cos kx, \quad \phi_a^{in} = i \sin kx. \quad (38)$$

In the outer region, we write

$$\phi_s^{sc} = D_s e^{ik|x|}, \quad \phi_a^{sc} = D_a \operatorname{sgn} x e^{ik|x|}, \quad (39)$$

where D_s and D_a are constants. Comparing with (6), we see that

$$R = D_s - D_a, \quad T - 1 = D_s + D_a. \quad (40)$$

(Note that if the condition (25) on R and T , arising from Lamb's method, is satisfied, then D_s must vanish, i.e. no symmetrical wave-like disturbance is permissible. This would be reasonable if the forcing were purely antisymmetric, as in the laboratory experiment conducted by Lamb & Cook (1910). It is also true when Γ is a thin barrier, i.e. when the grating is composed of a periodic row of apertures in a rigid screen along the y -axis.)

The inner expansions of the outer solutions, (38) and (39), are

$$\phi_a = \phi_a^{sc} + \phi_a^{in} \sim D_a \operatorname{sgn} x + (1 + D_a) ikx \quad (41)$$

and
$$\phi_s = \phi_s^{sc} + \phi_s^{in} \sim (1 + D_s) + ik|x| D_s - \frac{1}{2}(1 + D_s) k^2 x^2 \quad (42)$$

as $k|x| \rightarrow 0$. We shall determine D_a by matching with ψ_a , the inner solution from before. Note that, for D_s , we need three terms in (42), as explained later.

2.2.1. Determination of D_a

For an inner solution, we want an antisymmetric function Φ_a that satisfies all the boundary conditions and the differential equation (2) approximately for small kh . The harmonic function ψ_a , used earlier, satisfies this requirement and so we write

$$\Phi_a = A_a \psi_a(x, y), \quad (43)$$

where A_a is a constant to be determined. There is no additive constant in this solution since a constant would be symmetric. The outer expansion of Φ_a is given by

$$\Phi_a \sim \frac{(x + l \operatorname{sgn} x) A_a}{h}, \quad (44)$$

where we have used (31). Matching (41) and (44) gives

$$D_a = \frac{A_a l}{h}, \quad (1 + D_a) ik = \frac{A_a}{h}, \quad (45)$$

whence

$$D_a = \frac{ikl}{1 - ikl}, \quad A_a = \frac{ikh}{1 - ikl}. \quad (46)$$

2.2.2. Determination of D_s

By analogy with the antisymmetric problem, we are encouraged to look for a symmetric harmonic function Φ_s which satisfies the boundary conditions and $\sim x^2/(2h^2)$ as $|x| \rightarrow \infty$. However, a simple application of Green's theorem shows that this problem has no solution, as there is fluid flowing outwards in both directions, yet there are no sources.

Instead, let us assume that we can write our symmetric inner solution as

$$\Phi_s = A_s \psi_s + B, \tag{47}$$

where A_s and B are constants to be determined. Substituting Φ_s into (2) gives

$$h^2 \nabla^2 \psi_s + (kh)^2 \psi_s + (kh)^2 \frac{B}{A_s} = 0. \tag{48}$$

For $kh \ll 1$, we may discard the middle term in this equation, but not necessarily the last term, involving the constant B , since B/A_s will be a function of kh . Indeed, *a posteriori*, we show that $A_s = O((kh)^2)$ as $kh \rightarrow 0$; see (62). So without loss of generality, we require ψ_s to satisfy the Poisson equation

$$h^2 \nabla^2 \psi_s = 1 \text{ in the fluid,} \tag{49}$$

$$(\psi_s)_n = 0 \text{ on the boundaries,} \tag{50}$$

$$\psi_s \sim \frac{x^2}{2h^2} \text{ as } |x| \rightarrow \infty. \tag{51}$$

It is convenient to set
$$\psi_s = \frac{x^2}{2h^2} + \chi_s, \tag{52}$$

where χ_s is a harmonic function, which satisfies the following conditions:

$$(\chi_s)_n = -\left(\frac{x^2}{2h^2}\right)_n \text{ on the boundaries,} \tag{53}$$

$$\chi_s \sim M \frac{|x|}{h} + o(1) \text{ as } |x| \rightarrow \infty. \tag{54}$$

The condition (54) ensures uniqueness by eliminating arbitrary additive constants. Physically, χ_s corresponds to fluid flowing into the domain from both infinities with flux $|M|$, and then flowing out through the boundary Γ as the outflow vanishes on the other boundaries. Conservation of mass (using Green's theorem) implies that

$$2M = \int_{\Gamma} (\chi_s)_n ds = - \int_{\Gamma} \left(\frac{x^2}{2h^2}\right)_n ds \tag{55}$$

$$= - \iint_{D_-} \nabla^2 \left(\frac{x^2}{2h^2}\right) dS = -\frac{S}{h^2}, \tag{56}$$

where S is the cross-sectional area of the protrusion Γ (i.e. the area of D_-). Therefore,

$$M = -\frac{S}{2h^2}. \tag{57}$$

If Γ is a depression, we obtain

$$M = \frac{S}{2h^2} \quad (58)$$

after proper consideration of the normal directions. These results, (57) and (58), are exact.

The outer expansion of Φ_s , (47), is

$$\Phi_s \sim \frac{(\frac{1}{2}x^2 + m|x|)A_s}{h^2} + B, \quad (59)$$

where m is defined as

$$m = Mh. \quad (60)$$

Matching (59) and (42) gives

$$(1 + D_s) = B, \quad ikD_s = \frac{mA_s}{h^2}, \quad -\frac{1}{2}(1 + D_s)k^2 = \frac{A_s}{2h^2}, \quad (61)$$

which yields

$$D_s = \frac{ikm}{1 - ikm}, \quad A_s = -\frac{(kh)^2}{1 - ikm}. \quad (62)$$

(The coefficient A_s is small, $O((kh)^2)$, as required.)

The three terms were needed in (42) because an arbitrary constant B is allowed in the solution of the symmetric inner problem; thus, we need three conditions to determine the three constants, D_s , A_s and B .

2.3. Reflection and transmission coefficients

If we substitute D_s and D_a into the definitions of R and T , (40), we obtain

$$R = \frac{ik(m-l)}{(1-ikl)(1-ikm)}, \quad T = \frac{1+k^2lm}{(1-ikl)(1-ikm)}. \quad (63)$$

These formulae satisfy the energy conditions, (9) and (10), identically. They reduce to Lamb's formulae (23), when $m = 0$, i.e. when Γ represents a thin barrier on the y -axis.

For small circular cylinders of radius a we have

$$l \approx \frac{\pi a^2}{2h}, \quad m = -\frac{\pi a^2}{4h}, \quad (64)$$

where we have used (20), (24), (57) and (60). From (63),

$$R \approx ik(m-l) \approx -\frac{3\pi ka^2}{4h} \quad (65)$$

and

$$T \approx 1 + ik(l+m) \approx 1 + \frac{i\pi ka^2}{4h}, \quad (66)$$

in agreement with the known correct approximation (34).

3. Discussion and applications

The method described above is unusual in that the symmetric inner problem is not governed by the Laplace (or the Helmholtz) equation, but the Poisson equation. This arises because we need a function that grows like x^2 to match with the last term in (42), but there is no harmonic function that grows in this fashion and satisfies all the

boundary conditions (even when the protrusion is absent; x^2 is not harmonic.) For a more systematic application of the matched-asymptotic-expansions method, see Appendix D.

3.1. Circular cylinders

Equations (63) agree with the known exact results for small circular cylinders of radius a . In this case, we can also find an explicit approximation to ψ_s , the inner symmetric solution:

$$\tilde{\psi}_s(x, y) = \frac{x^2}{2h^2} - \frac{1}{4} \left(\frac{a}{h}\right)^2 \log \left[2 \left(\cosh \left(\frac{\pi x}{h} \right) - \cos \left(\frac{\pi y}{h} \right) \right) \right]. \tag{67}$$

This function satisfies $h^2 \nabla^2 \tilde{\psi}_s = 1$ in the fluid, (68)

$$(\tilde{\psi}_s)_y = 0 \text{ on the plane walls,} \tag{69}$$

$$\int_{\Gamma} (\tilde{\psi}_s)_n = 0. \tag{70}$$

Moreover, $\tilde{\psi}_s \sim \frac{1}{2} \left(\frac{x}{h}\right)^2 - \frac{1}{4} \pi \left(\frac{a}{h}\right)^2 \frac{|x|}{h} + o(1)$ as $|x| \rightarrow \infty$, (71)

whence, from (54) and (60), we see that we have agreement with the exact value for m , given by (64). Thus, $\tilde{\psi}_s$ is an approximation to ψ_s in that the boundary condition on Γ is only satisfied on average, (70).

For larger circular cylinders, we can still use (63), with m given by (64), provided we can compute the blockage coefficient l . An approximation to l can be obtained by using a conformal mapping due to Richmond (1923); see also Smythe (1968, §4.28). This approximation is (see Appendix C)

$$l \approx \frac{\alpha}{\beta} \log (\sec \beta), \tag{72}$$

where β depends on a/h and is a solution to the following equation:

$$\left(2\beta - \frac{\pi a}{h} \right) \log (\sec \beta + \tan \beta) = \frac{\pi a \beta}{h}. \tag{73}$$

For small β , this equation gives $\beta \approx \frac{\pi a}{h}$. (74)

Thus, for small a/h , (72) gives

$$l \approx \frac{a}{\beta} \log \left(1 + \frac{1}{2} \beta^2 \right) \approx \frac{1}{2} a \beta \approx \frac{\pi a^2}{2h}, \tag{75}$$

in agreement with (64). Table 1 provides values of β for various values of a/h , together with values of l/h computed from (72) and from the approximation, (64). From the table, the small-cylinder approximation (64) underestimates Richmond's values by over 10% for $a/h > 1/3$.

3.2. Small elliptical cylinders

We can reverse the above procedure and obtain an approximation for l for small elliptical protrusions by comparing (63) with the exact results derived by Burke & Twersky (1966). For an ellipse defined by

$$\left(\frac{x}{b}\right)^2 + \left(\frac{y}{a}\right)^2 = 1, \tag{76}$$

| a/h | β | l/h | $\frac{1}{2}\pi(a/h)^2$ |
|-------|-----------|--------|-------------------------|
| 0.1 | 0.3116 | 0.0158 | 0.0157 |
| 0.2 | 0.6083 | 0.0650 | 0.0628 |
| 0.3 | 0.8774 | 0.1530 | 0.1414 |
| 0.4 | 1.1095 | 0.2918 | 0.2513 |
| 0.5 | 1.2985 | 0.5056 | 0.3927 |
| 0.6 | 1.4402 | 0.8492 | 0.5655 |
| 0.7 | 1.5305 | 1.4692 | 0.7697 |
| 0.8 | 1.5673 | 2.8825 | 1.0053 |
| 0.9 | 1.5707957 | 8.1833 | 1.2723 |

TABLE 1. The blockage coefficient for an approximately semicircular protrusion of radius a in a channel of width h , computed using Richmond's conformal mapping (73). The fourth column gives the approximation (64) derived for small a/h .

they obtain (their equation 63)

$$R \sim -\frac{i\pi ka}{4h}(a+2b), \quad T \sim \frac{i\pi ka^2}{4h} \quad (77)$$

for long waves and small cylinders. From (63), we have

$$R \sim -ik(l-m), \quad T \sim 1+ik(l+m). \quad (78)$$

Comparing (77) and (78), we can solve for l and m to give

$$l = \frac{\pi a}{4h}(a+b) \quad (79)$$

and

$$m = -\frac{\pi ab}{4h}. \quad (80)$$

This equation for m agrees with the exact result, given by (57) and (60). The expression for l , (79), reduces to (64) for circles ($a = b$), vanishes when $a = 0$ (i.e. when the ellipse vanishes) and gives

$$l = \frac{\pi a^2}{4h}, \quad (81)$$

when $b = 0$ (i.e. when the ellipse degenerates into a small vertical barrier of height a). The approximation (81) also agrees with the exact result for a vertical barrier,

$$l = \frac{2h}{\pi} \log \left\{ \sec \left(\frac{\pi a}{2h} \right) \right\} \quad (82)$$

as given by Lamb (1932, §306). These results will be used in §5.

3.3. Small cylinders of arbitrary symmetric cross-section

Miles (1982a) has given formulae similar to (63) for the scattering of long waves by a grating of small vertical cylinders. He considers asymmetric cylinders and oblique incidence. If we specialize his results to symmetric cylinders and normal incidence, we obtain

$$R = \frac{ikr}{1 - ikt + \frac{1}{4}k^2(r^2 - t^2)}, \quad T = \frac{1 - \frac{1}{4}k^2(r^2 - t^2)}{1 - ikt + \frac{1}{4}k^2(r^2 - t^2)}, \quad (83)$$

where

$$r = -\frac{1}{4h}(4S + \mathcal{A}), \quad t = \frac{1}{4h}\mathcal{A}. \tag{84}$$

The quantity \mathcal{A} is the added mass of a *single cylinder* (Γ and its reflection in $y = 0$), translating with constant speed U in the x -direction in an otherwise unbounded incompressible fluid of unit density. The fluid is at rest at ∞ , and its kinetic energy per unit length is $\frac{1}{2}U^2\mathcal{A}$; explicitly,

$$\mathcal{A} = 2 \int_{\Gamma} \phi_{\infty} x_n \, ds, \tag{85}$$

where $\nabla^2\phi_{\infty} = 0$ in the fluid, $(\phi_{\infty} + x)_n = 0$ on the cylinder and $|\phi_{\infty}| \rightarrow 0$ at ∞ (Lamb 1932, §121).

If we compare Miles’s result (83) with our (63), we find that they agree precisely if

$$l = \frac{1}{4h}(2S + \mathcal{A}), \tag{86}$$

after using (57) and (60). This is an approximation to the blockage coefficient l , valid for *small* protrusions. For the ellipse, (76), $\mathcal{A} = \pi a^2$, $2S = \pi ab$, and (86) reduces to (81). Note that our formulae (63) do not require that the protrusion be small.

The above approximation (86) is consistent with a known exact result, connecting l to the added mass of the cylinder in a channel, \mathcal{A}_c :

$$l = \frac{1}{4h}(2S + \mathcal{A}_c). \tag{87}$$

This equation is given by Sedov (1965, p. 146, eq. 8.6) and Newman (1969, eq. 3.4), and will be used in §5.3. In our notation, we have (cf. (85))

$$\mathcal{A}_c = 2 \int_{\Gamma} (h\psi_a - x) x_n \, ds. \tag{88}$$

4. Scattering of water waves by horizontal cylinders

The matched-asymptotic methods discussed in the previous sections on acoustic waves will now be applied to water waves impinging on a long horizontal cylinder fixed in the free surface ($y = 0$) or at the bottom ($y = h$). (For convenience, cylinders fixed in the free surface are termed ‘floating’ below). The cylinders are in a channel of constant (finite) depth h . As before, the surface of the cylindrical obstacle will be denoted by Γ , which is assumed to be symmetric about $x = 0$. The cylinder has width $2b$: if it is floating, L denotes the length of the mean water level intersected by the cylinder ($y = 0$, $|x| \leq b$); if it is a ridge or a trench, L is a segment, $y = h$, $|x| \leq b$.

The incident wave will have a velocity potential

$$\phi^{\text{in}}(x, y) = Y(y) e^{ikx}, \tag{89}$$

where

$$Y(y) = \frac{\cosh k(h-y)}{\cosh kh}, \tag{90}$$

k is the unique positive real root of

$$K = k \tanh kh, \tag{91}$$

$K = \omega^2/g$ and g is the acceleration due to gravity. The total velocity potential ϕ satisfies

$$\nabla^2\phi = 0 \text{ in the fluid,} \tag{92}$$

$$K\phi + \phi_y = 0 \text{ on the mean free surface } (y = 0), \tag{93}$$

$$\phi_n = 0 \text{ on all rigid boundaries,} \tag{94}$$

$$\phi^{sc} \equiv \phi - \phi^{in} \text{ satisfies a radiation condition as } |x| \rightarrow \infty. \tag{95}$$

We write the radiation condition, (95), as

$$\phi \sim \begin{cases} Y(y)T e^{ikx} & \text{as } x \rightarrow \infty, \\ Y(y)(e^{ikx} + R e^{ikx}) & \text{as } x \rightarrow -\infty, \end{cases}$$

or, equivalently, as
$$\phi^{sc} \sim \begin{cases} Y(y)(T-1) e^{ikx} & \text{as } x \rightarrow \infty, \\ Y(y)R e^{ikx} & \text{as } x \rightarrow -\infty. \end{cases}$$

As before, the reflection coefficient R and the transmission coefficient T must satisfy (7) and (8) (or (9) and (10)); see e.g. Newman (1977) or Martin (1985).

4.1 Simple matching procedure for shallow-water waves

The second method, described in §2.1, has been used to treat the water-wave problem posed above, for the case of long waves; see Guiney *et al.* (1972), Tuck (1977), and Miles (1982*b*). The procedure is reviewed below.

Assume that Kb and Kh are both small; from (91), we have

$$kh \approx (Kh)^{\frac{1}{2}} \ll 1. \tag{96}$$

Therefore, from (90), $Y(y) \approx 1$. So, we take (26) as an outer solution; its inner expansion is (28), namely

$$\phi \sim \begin{cases} T + ikxT, & x > 0, \\ 1 + R + ikx(1 - R), & x < 0, \end{cases} \tag{97}$$

as $k|x| \rightarrow 0$. Let Φ be an inner solution, satisfying (92) and (94). Since (96) holds, we can replace the free-surface condition, (93), by the ‘rigid-lid’ condition

$$\Phi_y = 0 \text{ on } y = 0. \tag{98}$$

In order to match Φ with (97), assume that

$$\Phi \sim \Phi_{\pm} + x\Phi'_{\pm} \text{ as } \frac{|x|}{h} \rightarrow \infty, \tag{99}$$

where Φ_{\pm}, Φ'_{\pm} do not depend on x . Since all the boundaries are rigid, conservation of mass implies that

$$\Phi'_+ = \Phi'_- = A, \tag{100}$$

where A is a constant. Thus, the inner solution corresponds to potential flow past an obstacle, Γ , in a channel; therefore,

$$\Phi = A\psi_a + B,$$

where ψ_a is the inner solution discussed in §2.1, and B is an arbitrary constant; in particular,

$$\Phi \sim \frac{(x + l \operatorname{sgn} x)A}{h} + B \text{ as } |x| \rightarrow \infty.$$

Matching this with (97), we obtain

$$R = 1 - T = -\frac{ikl}{1 - ikl}$$

which is (23). We observe that this formula does not distinguish between a floating cylinder and a cylindrical bottom ridge, when both have the same cross-section.

The argument used above is incomplete. The result is correct for submerged ridges and trenches, but not for floating cylinders. One defect is in the imposition of the ‘rigid-lid’ condition, (98), for this implies (100), which in turn implies that $R + T = 1$; in general, this is too restrictive: (9) gives merely that $|R + T| = 1$. The rigid-lid condition is only one approximation to the exact free-surface condition, (93). We shall exploit this observation in the next section. Again, see Appendix D for a more formal derivation.

4.2. Modified matching procedure

As before, we separate the scattering problem into two problems, one symmetric and one antisymmetric about $x = 0$. We have

$$\phi_a^{\text{in}} = iY(y) \sin kx, \quad \phi_s^{\text{in}} = Y(y) \cos kx. \quad (101)$$

In the outer region, we write

$$\phi_a^{\text{sc}} = D_a Y(y) \operatorname{sgn} x e^{ik|x|}, \quad \phi_s^{\text{sc}} = D_s Y(y) e^{ik|x|}. \quad (102)$$

Comparing with (4), we see that R and T are given by (40). The inner expansion of the outer solutions are

$$\phi_a \sim Y(y) [D_a \operatorname{sgn} x + (1 + D_a) ikx], \quad (103)$$

$$\phi_s \sim Y(y) [(1 + D_s) + ik|x|D_s - \frac{1}{2}(1 + D_s)k^2x^2] \quad (104)$$

as $k|x| \rightarrow 0$; for small kh , these simplify further to

$$\phi_a \sim D_a \operatorname{sgn} x + (1 + D_a) ikx, \quad (105)$$

$$\phi_s \sim (1 + D_s) + ik|x|D_s - \frac{1}{2}k^2(1 + D_s)(x^2 - y^2 + 2hy). \quad (106)$$

4.2.1. Determination of D_a

The antisymmetric inner solution is Φ_a , given by $\Phi_a = A_a \psi_a$, where ψ_a is the harmonic function defined in §2.1. Matching with (105), we obtain

$$D_a = \frac{ikl}{1 - ikl}, \quad (107)$$

as before, where l is the blockage coefficient.

4.2.2. Determination of D_s

We now assume that we can write our symmetric inner solution as

$$\Phi_s = A_s \hat{\psi}_s + B. \quad (108)$$

We require Φ_s to satisfy (92) and (94). The free-surface condition (93) gives

$$Kh\hat{\psi}_s + h(\hat{\psi}_s)_y + \frac{KhB}{A_s} = 0. \quad (109)$$

For shallow-water waves, (96), we can neglect the first term. The last term may not be negligible; in fact, if the last term is $O(1)$, then the rigid-lid approximation (98) is invalid, and must be modified to include the surface velocity induced by the presence of the cylinder. So we require $\hat{\psi}_s$ to satisfy

$$h(\hat{\psi}_s)_y = 1 \text{ on the free surface.} \tag{110}$$

It is convenient to write $\hat{\psi}_s$ as

$$\hat{\psi}_s = \chi_s^0 + \chi_s^1, \tag{111}$$

where

$$\chi_s^0 = \frac{(x^2 - y^2 + 2hy)}{2h^2}, \tag{112}$$

and χ_s^1 satisfies Laplace's equation in the fluid and the following boundary conditions:

$$(\chi_s^1)_y = 0 \text{ on both the free surface and the flat bottom,} \tag{113}$$

$$(\chi_s^1)_n = -(\chi_s^0)_n \text{ on } \Gamma, \tag{114}$$

$$\chi_s^1 \sim \frac{N|x|}{h} + o(1) \text{ as } |x| \rightarrow \infty. \tag{115}$$

We can find N exactly, using Green's theorem:

$$2N = \int_{\Gamma} (\chi_s^1)_n ds = - \int_{\Gamma} (\chi_s^0)_n ds = - \int_L (\chi_s^0)_n dx, \tag{116}$$

since χ_s^0 is harmonic. There are now two cases.

(i) If Γ corresponds to a ridge or a trench, L is part of $y = h$, on which

$$(\chi_s^0)_n = -(\chi_s^0)_y = 0. \tag{117}$$

Therefore, $N = 0$. The same result obtains if Γ is a closed curve (submerged cylinder).

(ii) If Γ corresponds to a floating cylinder, L is part of $y = 0$, on which

$$(\chi_s^0)_n = (\chi_s^0)_y = \frac{1}{h}, \tag{118}$$

giving

$$N = -\frac{b}{h}, \tag{119}$$

where $2b$ is the width of the cylinder at $y = 0$ (this is the length of L).

The outer expansion of Φ_s , (108), is

$$\Phi_s \sim \frac{\{\frac{1}{2}(x^2 - y^2 + 2hy) + n|x|\} A_s + B}{h^2}, \tag{120}$$

where we have defined n by

$$n = Nh. \tag{121}$$

Matching (106) and (120) gives

$$1 + D_s = B, \quad ikD_s = \frac{nA_s}{h^2}, \quad -\frac{1}{2}(1 + D_s)k^2 = \frac{A_s}{2h^2}.$$

Hence,

$$D_s = \frac{ikn}{1 - ikn}, \quad A_s = -\frac{(kh)^2}{1 - ikn}. \tag{122}$$

Note that A_s is small, as required; also, for ridges and trenches $D_s = 0$, whereas D_s does not vanish for floating cylinders.

4.3. Reflection and transmission coefficients

Substituting (107) and (122) into (40), we obtain

$$R = \frac{ik(n-l)}{(1-ikl)(1-ikn)}, \quad T = \frac{1+k^2ln}{(1-ikl)(1-ikn)}. \quad (123)$$

These formulae satisfy the energy conditions, (9) and (10), identically. Furthermore, as the waves become very long, R and T approach zero and unity, respectively, which is expected for very low-frequency motions, which would have time to completely pass the obstacle over a wave period.

For ridges and trenches, (117) and (121) show that $n = 0$, so that (123) reduce to

$$R = -\frac{ikl}{1-ikl}, \quad T = \frac{1}{1-ikl}. \quad (124)$$

These formulae were also obtained by Miles (1982*b*). If kl is small, these coefficients can be further approximated by

$$R \approx -ikl, \quad T \approx 1+ikl. \quad (125)$$

These last approximations were given by Tuck (1977) and Miles (1982*b*); the approximation $|R| \approx k|l|$ was earlier obtained by Kreisel (1949).

For floating cylinders, $n = -b$, whence (123) reduce to

$$R = -\frac{ik(l+b)}{(1-ikl)(1+ikb)}, \quad T = \frac{1-k^2lb}{(1-ikl)(1+ikb)}. \quad (126)$$

These approximations are new. For small cylinders, they give

$$R \approx -ik(l+b), \quad T \approx 1+ik(l-b). \quad (127)$$

In the next section, we give some independent verification of the reflection and transmission coefficients given by (123).

5. Discussion, verification and applications

For ridges and trenches, our approximations for R and T , (123), reduce to formulae obtained previously by other authors (who have used incomplete arguments). Our formulae were derived under the assumptions that $Kb \ll 1$ and $Kh \ll 1$. Also, we implicitly assumed that if the obstacle was a trench, then it was not too deep (because the free-surface condition above the trench was approximated). Under these assumptions, it is known that the formulae for R and T agree well with exact solutions; see e.g. Kirby & Dalrymple (1983) for rectangular trenches. For thin vertical barriers, whether surface piercing ($b = 0$) or not, we also obtain (124). Tuck (1975, §VIII.C) has verified that these agree with the exact solution by Packham & Williams (1972).

We now consider three special geometries, corresponding to semicircular, rectangular, and elliptical cylinders.

5.1. Half-immersed circular cylinder and semicircular ridge

For small semicircular obstacles of radius a we can use a hydrodynamic analogue of Lewin's 'multiplet' theory (Appendix A). In this theory, the scattered waves are approximated by

$$\phi^{\text{sc}} = KhA_1 \tilde{\Phi}_1 + \kappa hA_2 \tilde{\Phi}_2, \quad (128)$$

where $\tilde{\Phi}_1(\tilde{\Phi}_2)$ is the potential due to a wave source (horizontal wave dipole) at the centre of the semicircle, and A_1 and A_2 are constants to be determined by imposing the boundary condition (94) on Γ . When this is done, we obtain (124) for the ridge, and (126) (with $b = a$) for the floating cylinder; in each case, $l = \frac{1}{2}\pi a^2/h$. See Appendix B for details.

We can also compare our results with the numerical results of Naftzger & Chakrabarti (1979), obtained by solving an integral equation over Γ (in principle, their results are exact). They give graphs of $|R|$ as a function of ka , for several values of a/h and for both the ridge and the floating cylinder. For example, when $a/h = 0.25$ and $ka = 0.1$, we can estimate from their figures 5 and 14 that $|R| \approx 0.14$ for a floating cylinder and $|R| \approx 0.04$ for a ridge. According to our approximations ((125) and (127)), these two numbers should differ by ka , as they do. Moreover, for the ridge, (125) gives

$$|R| \approx kl \approx \frac{\pi ka^2}{2h} \approx 0.04,$$

in agreement with the graphical estimate.

For larger cylinders, we can use the approximation to l given by (72). In particular, for $a/h = 0.5$ and $ka = 0.1$, we obtain $kl \approx 0.11$ from table 1, whence $|R| \approx 0.21$ for a floating cylinder and $|R| \approx 0.11$ for a submerged ridge. The corresponding estimates from Naftzger & Chakrabarti (1979) are $|R| \approx 0.19$ and $|R| \approx 0.11$.

5.2. Half-immersed elliptical cylinder and semi-elliptical ridge

For small semi-elliptical obstacles, of width $2b$ and height a , the blockage coefficient l is given by (79), whence (125) gives

$$R \sim -\frac{ik\pi a}{4h}(a+b) \quad (129)$$

for a ridge, whereas (127) yields

$$R \sim -\frac{ik\pi a}{4h}(a+b) - ikb \quad (130)$$

for a floating cylinder. In particular, for vertical barriers of height a ($b = 0$) both formulae reduce to

$$R \sim -\frac{ik\pi a^2}{4h}. \quad (131)$$

When $a = 0$, the floating cylinder reduces to a finite dock, of width $2b$. In this case, (130) gives

$$R \approx -ikb. \quad (132)$$

In fact, $l = 0$, and so (126) yields

$$R \approx -\frac{ikb}{1+ikb}, \quad T \approx \frac{1}{1+ikb}. \quad (133)$$

This problem has been solved by Stoker (1957, pp. 430–433), using shallow-water theory. He obtains

$$R \approx -\frac{ikb e^{-2ikb}}{1-ikb}, \quad T \approx \frac{e^{-2ikb}}{1-ikb}. \quad (134)$$

It is easily verified that, for small kb , (133) and (134) agree with an error of $O((kb)^3)$.

5.3. One rectangular obstacle

Mei & Black (1969) have obtained results for waves impinging on rectangular cylinders, using Schwinger's variational method and they have given graphs of $|R|$ as a function of a dimensionless wavenumber, for several geometries.

For a rectangle with width $2b$ and a height d , the blockage coefficient l is given by (88) (with $\mathcal{A}_c = \lambda_x/\rho$) as

$$\frac{lh}{bd} = 1 + \frac{\lambda_x}{4\rho bd}, \quad (135)$$

where ρ is the density of the fluid and $\lambda_x/(4\rho bd)$ is the added-mass coefficient for the rectangle (see e.g. Taylor 1973). This coefficient has been tabulated by Flagg & Newman (1971) for various values of b/d and d/h .

Two comparisons will be made. For a rectangular ridge, let $H = h - d$, the depth of water above the ridge. For $b = 2H$ and $h = 2H$, we can estimate from figure 2 of Mei & Black (1969) that

$$|R| \sim 2.5kH \quad \text{as } kH \rightarrow 0. \quad (136)$$

From (125), we predict that

$$|R| \sim kl = kH \left(\frac{l}{H} \right) \quad \text{as } kH \rightarrow 0. \quad (137)$$

Now, since $b/d = 2$, $d/h = \frac{1}{2}$ and $h/H = 2$, we have $l/H = lh/bd = 2.4997$, where we have used (135) and table 1 from Flagg & Newman (1971). Thus, we have good agreement.

For a floating rectangular cylinder, Mei & Black (1969) have plotted $|R|$ as a function of kd ($\equiv kH$); for $b = d$ and $h = 2d$, we estimate that from their figure 6 that

$$|R| \sim 2.5kd \quad \text{as } kd \rightarrow 0. \quad (138)$$

From (127), we predict that

$$|R| \sim k(l+b) = \frac{kd(l+b)}{d} \quad \text{as } kd \rightarrow 0. \quad (139)$$

Now, again using table 1 from Flagg & Newman (1971), we have

$$\frac{l+b}{d} = \left(\frac{lh}{bd} \right) \left(\frac{b}{h} \right) + \frac{b}{d} = \frac{1}{2}(1.9992 + 1) + 1 \approx 2.5. \quad (140)$$

It is clear that (138) agrees very well with (139).

For a very small gap between a floating rectangular object of width $2b$ and the bottom, the asymptotic formula of Flagg & Newman for the blockage coefficient can be used:

$$l \approx \frac{b}{\epsilon} + \frac{2h}{\pi} - b - \frac{2h}{\pi} \log 4\epsilon + O(\epsilon^2), \quad (141)$$

where $h\epsilon$ is the gap width. Using only the leading term in (141) for a small gap, our formulae (123) for the reflection and transmission coefficients give

$$R = -\frac{ikb(1+\epsilon)}{(\epsilon-ikb)(1+ikb)}, \quad T = \frac{\epsilon-k^2b^2}{(\epsilon-ikb)(1+ikb)}. \quad (142)$$

5.4. An aperture in a thick wall

Guiney *et al.* (1972) have considered the transmission of waves through a small aperture in a thick wall, for both deep and shallow water. In the latter case, for a thick, surface-piercing barrier, they have incorrectly obtained Lamb's formulae (23). Our formulae (123) are applicable to an arbitrary-shaped symmetric (in x) aperture. If we consider the case of a bottom-mounted cylinder directly beneath an identical inverted surface-piercing cylinder, then the blockage coefficient is the same as for a single cylinder in a channel of width $\frac{1}{2}h$. For a pair of rectangular cylinders, each of width $2b$ and height $\frac{1}{2}d$, separated by a gap of $h-d$, we can compute l using the results in Flag & Newman (1971). For a small gap, we can use the leading term in the asymptotic approximation for l , (141), and we find the reflection and transmission coefficients given in (142). Thus, for shallow-water waves, the reflection and transmission coefficients for a gap of width $h-d$ are asymptotically the same for the gap placed in the middle of the wall as they are for the gap at the bottom of the wall.

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Appendix A. An approximate theory for scattering from a cylindrical grating

A row of circular cylinders is located along the y -axis. The centre of the n th cylinder is at O_n , with coordinates $x = 0$, $y = 2nh$. Assume that we can write

$$\phi^{\text{sc}} \approx A_0 \sum_{n=-\infty}^{\infty} H_0^{(1)}(kR_n) + iA_1 \sum_{n=-\infty}^{\infty} H_1^{(1)}(kR_n) \cos \theta_n, \quad (A 1)$$

where $x = R_n \cos \theta_n$, $y - 2nh = R_n \sin \theta_n$, (R_n and θ_n are polar coordinates centred on O_n), and A_0 and A_1 are constants to be determined. Thus, we have approximated ϕ^{sc} throughout the fluid domain by identical wave sources and x -directed dipoles at O_n , $n = 0, \pm 1, \pm 2, \dots$. From Twersky (1961), we can write (A 1) as an infinite series of plane waves, some of which are propagating and some of which are evanescent as $|x| \rightarrow \infty$. In particular, if $kh < \pi$, there is only one propagating wave and, if $|x|$ is also large, we obtain

$$\phi^{\text{sc}} \sim (kh)^{-1} \{A_0 + A_1 \operatorname{sgn} x\} e^{ik|x|} \quad \text{as } |x| \rightarrow \infty. \quad (A 2)$$

Comparing with (6), we see that

$$R \approx \frac{A_0 - A_1}{kh}, \quad T - 1 \approx \frac{A_0 + A_1}{kh}. \quad (A 3)$$

We now determine A_0 and A_1 by imposing the no-flow boundary condition (3) on the cylinder. We assume that the potential near any cylinder is approximately due to the source and dipole at its centre *only*; it is here that we use the fact that $a/h \ll 1$. Near the central cylinder ($n = 0$), we have

$$\phi^{\text{sc}} \approx A_0 H_0(kR) + iA_1 H_1(kR) \cos \theta, \quad (\text{A } 4)$$

where $R \equiv R_0$ and $\theta \equiv \theta_0$. Also, the incident wave is given by

$$\phi^{\text{in}} = e^{ikx} = \sum_{n=0}^{\infty} \epsilon_n i^n J_n(kR) \cos n\theta \quad (\text{A } 5)$$

$$\approx J_0(kR) + 2iJ_1(kR) \cos \theta. \quad (\text{A } 6)$$

We require that

$$(\phi^{\text{sc}} + \phi^{\text{in}})_r = 0 \quad \text{on } R = a \quad (\text{A } 7)$$

from which we obtain the following two relationships:

$$A_0 H'_0(ka) + J'_0(ka) = 0, \quad (\text{A } 8)$$

$$A_1 H'_1(ka) + 2J'_1(ka) = 0. \quad (\text{A } 9)$$

For small ka , we can approximate the Bessel and Hankel functions to give

$$A_0 \approx -\frac{1}{4}i\pi(ka)^2, \quad A_1 \approx \frac{1}{2}i\pi(ka)^2.$$

When these are substituted into (A 3), we obtain (34).

Appendix B. Semicircular obstacles: simple multipole approximations

A Green function for the velocity potential at (x, y) due to a wave source at (ξ, η) in water of depth h is (see e.g. Ursell 1981).

$$G(x, y; \xi, \eta) = \frac{1}{2} \log \frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2} - 2 \int_0^{\infty} e^{-\kappa h} \frac{\sinh \kappa y \sinh \kappa \eta}{\kappa \cosh \kappa h} \cos \kappa(x-\xi) \, d\kappa \\ - 2 \int_0^{\infty} \frac{\cosh \kappa(h-y) \cosh \kappa(h-\eta) \cos \kappa(x-\xi)}{\cosh \kappa h (\kappa \sinh \kappa h - K \cosh \kappa h)} \, d\kappa, \quad (\text{B } 1)$$

where the contour is indented below the simple pole of the integrand at $\kappa = k$, and k is the unique positive real root of (91). For fixed ξ ,

$$G \sim -2\pi i E Y(y) Y(\eta) e^{ik|x-\xi|} \quad \text{as } |x| \rightarrow \infty, \quad (\text{B } 2)$$

where $Y(y)$ is defined by (90) and

$$E = \frac{2 \cosh^2 kh}{2kh + \sinh 2kh}.$$

B.1. Elementary solutions

We shall need to use some simpler functions derived from the Green function.

Wave source at origin (0, 0)

$$\text{Define } \tilde{\Phi}_2 = -\frac{1}{2}G(x, y; 0, 0) \quad (\text{B } 3)$$

$$= \int_0^{\infty} \frac{\cosh \kappa(h-y) \cos \kappa x}{\kappa \sinh \kappa h - K \cosh \kappa h} \, d\kappa \quad (\text{B } 4)$$

$$\sim \pi i E Y(y) e^{ik|x|} \quad \text{as } |x| \rightarrow \infty. \quad (\text{B } 5)$$

For small Kh and r/h ($r^2 = x^2 + y^2$) we have (Ursell 1976)

$$\operatorname{Re}(\tilde{\Phi}_2) \sim -\log r. \tag{B 6}$$

Also,
$$\operatorname{Im}(\tilde{\Phi}_2) = \pi E Y(y) \cos kx.$$

Horizontal wave-dipole at (0, 0)

Define

$$\tilde{\Phi}_1 = -\frac{1}{K} (\tilde{\Phi}_2)_x \tag{B 7}$$

$$= \int_0^\infty \frac{\kappa \cosh \kappa(h-y) \sin \kappa x}{K(\kappa \sinh \kappa h - K \cosh \kappa h)} d\kappa + \pi i \frac{k}{K} E Y(y) \sin kx \tag{B 8}$$

$$\sim \pi \operatorname{sgn} x \frac{k}{K} E Y(y) e^{ik|x|} \quad \text{as } |x| \rightarrow \infty. \tag{B 9}$$

For small Kh and r/h , we have

$$\operatorname{Re}(\tilde{\Phi}_1) \sim \frac{\sin \theta}{Kr}, \tag{B 10}$$

where $x = r \sin \theta$, $y = r \cos \theta$. Also,

$$\operatorname{Im}(\tilde{\Phi}_1) = \pi \frac{k}{K} E Y(y) \sin kx. \tag{B 11}$$

Wave source at (0, h)

Define

$$\tilde{\Phi}_2^h(x, y) = -\frac{1}{2} G(x, y; 0, h) \tag{B 12}$$

$$= -\frac{1}{4} \log \frac{x^2 + (y-h)^2}{x^2 + (y+h)^2} + \int_0^\infty e^{-\kappa h} \frac{\sinh \kappa y \sinh \kappa h}{\kappa \cosh \kappa h} \cos \kappa x d\kappa + \int_0^\infty \frac{\cosh \kappa(h-y) \cos \kappa x}{\cosh \kappa h (\kappa \sinh \kappa h - K \cosh \kappa h)} d\kappa \tag{B 13}$$

$$\sim \pi i E \operatorname{sech} kh Y(y) e^{ik|x|} \quad \text{as } |x| \rightarrow \infty. \tag{B 14}$$

Since
$$\int_0^\infty e^{-\kappa h} \sinh \kappa y \frac{\cos \kappa x}{\kappa} d\kappa = -\frac{1}{4} \log \frac{x^2 + (y-h)^2}{x^2 + (y+h)^2} \quad \text{for } 0 < y < h, \tag{B 15}$$

we can write the second term as

$$-\frac{1}{4} \log \frac{x^2 + (y-h)^2}{x^2 + (y+h)^2} - \int_0^\infty e^{-2\kappa h} \frac{\sinh \kappa y}{\kappa \cosh \kappa h} \cos \kappa x d\kappa, \tag{B 16}$$

whence
$$\operatorname{Re}(\tilde{\Phi}_2^h) \sim -\log r_1 \tag{B 17}$$

for small Kh and r_1/h , where $r_1^2 = x^2 + (y-h)^2$. Also,

$$\operatorname{Im}(\tilde{\Phi}_2^h) = \pi E \operatorname{sech} kh Y(y) \cos kx. \tag{B 18}$$

Horizontal wave dipole at (0, h)

Define

$$\tilde{\Phi}_1^h = -\frac{1}{K} (\tilde{\Phi}_2^h)_x \tag{B 19}$$

$$\sim \operatorname{sgn} x \pi \frac{k}{K} E \operatorname{sech} kh Y(y) e^{ik|x|} \quad \text{as } |x| \rightarrow \infty. \tag{B 20}$$

For small Kh and r_1/h ,

$$\operatorname{Re}(\tilde{\Phi}_1^h) \sim \frac{\sin \theta_1}{Kr_1}, \quad (\text{B 21})$$

where $x = r_1 \sin \theta_1$, $h - y = r_1 \cos \theta_1$. Also,

$$\operatorname{Im}(\tilde{\Phi}_1^h) = \pi \frac{k}{K} E \operatorname{sech} khY(y) \sin kx. \quad (\text{B 22})$$

B.2. Half-immersed circular cylinder, radius a , centre $(0, 0)$

Using the above elementary solutions, we treat the case of a floating cylinder. Assume that the scattered potential can be written as

$$\phi^{\text{sc}} = KhA_1 \tilde{\Phi}_1 + \kappa hA_2 \tilde{\Phi}_2 \quad (\text{B 23})$$

$$\sim \pi khEY(y) (\operatorname{sgn} x A_1 + iA_2) e^{k|x|} \quad \text{as } |x| \rightarrow \infty, \quad (\text{B 24})$$

$$\text{whence, comparing with (4), } T - 1 = \pi khE(A_1 + iA_2) \quad (\text{B 25})$$

$$\text{and } R = \pi khE(-A_1 + iA_2). \quad (\text{B 26})$$

For small Kh , $E \sim (2kh)^{-1}$, whence

$$T - 1 \approx \frac{1}{2}\pi(A_1 + iA_2), \quad R \approx \frac{1}{2}\pi(-A_1 + iA_2). \quad (\text{B 27})$$

Now, to obtain the solution we have

$$\phi = \phi^{\text{sc}} + \phi^{\text{in}} \quad (\text{B 28})$$

$$\sim \frac{h}{r} A_1 \sin \theta - khA_2 \log r + i(\frac{1}{2}\pi A_1 + 1) Y \sin kx + (\frac{1}{2}\pi iA_2 + 1) Y \cos kx, \quad (\text{B 29})$$

$$\text{whence } \langle a\phi_r \rangle \sim -\frac{h}{a} A_1 \sin \theta - khA_2 + i(\frac{1}{2}\pi A_1 + 1) I_a(\theta) + (\frac{1}{2}\pi iA_2 + 1) I_s(\theta), \quad (\text{B 30})$$

$$\text{where } I_a(\theta) = \langle a(Y(y) \sin kx)_r \rangle, \quad I_s(\theta) = \langle a(Y(y) \cos kx)_r \rangle, \quad (\text{B 31})$$

and the angular brackets require setting $r = a$ after the differentiation with respect to r . We impose

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \langle a\phi_r \rangle d\theta = 0, \quad \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \langle a\phi_r \rangle \sin \theta d\theta = 0. \quad (\text{B 32})$$

$$\text{These equations give } -\pi khA_2 + (\frac{1}{2}\pi iA_2 + 1) \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} I_s(\theta) d\theta = 0 \quad (\text{B 33})$$

$$\text{and } -\frac{\pi h}{2a} A_1 + i(\frac{1}{2}\pi A_1 + 1) \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} I_a(\theta) \sin \theta d\theta = 0. \quad (\text{B 34})$$

Green's theorem shows that

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} I_s(\theta) d\theta = -\frac{2K}{k} \sin ka \approx -2Ka \approx -2k^2 ah, \quad (\text{B 35})$$

$$\text{whence (B 33) gives } \frac{1}{2}\pi iA_2 \approx -\frac{ika}{1 + ika}. \quad (\text{B 36})$$

$$\text{Since } I_a \sim ka \sin \theta, \text{ (B 34) gives } \frac{1}{2}\pi A_1 \approx \frac{ikl}{1 - ikl}, \quad (\text{B 37})$$

where l is the blockage coefficient for small circular cylinders, given by (20) and (24). Substituting (B 36) and (B 37) into (B 27), we obtain

$$R \approx -\frac{ikl}{1-ikl} - \frac{ika}{1+ika} \tag{B 38}$$

and

$$T-1 \approx \frac{ikl}{1-ikl} - \frac{ika}{1+ika} \tag{B 39}$$

in agreement with (123), for this special geometry.

B.3. Semicircular ridge, radius a , centre $(0, h)$

Assume that
$$\phi^{sc} = KhA_1^h \tilde{\Phi}_1^h + khA_2^h \tilde{\Phi}_2^h, \tag{B 40}$$

whence, for small Kh ,

$$T-1 \approx \frac{1}{2}\pi(A_1^h + iA_2^h), \quad R \approx \frac{1}{2}\pi(-A_1^h + iA_2^h). \tag{B 41}$$

We proceed as before, differentiating with respect to r_1 , setting $r_1 = a$ and imposing (B 32), integrating over θ_1 . The only difference occurs when we evaluate

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} a(Y(y) \cos kx)_{r_1} d\theta_1 = - \int_{-a}^a Y'(h) \cos kx dx = 0, \tag{B 42}$$

since $Y'(h) = 0$. Hence
$$A_1^h = A_1, \quad A_2^h = 0. \tag{B 43}$$

Substituting these into (B 41), we obtain the same formulae as (124), for this special geometry.

Appendix C. The blockage coefficient for large circular cylinders

Richmond (1923) has investigated the conformal mapping given by

$$\frac{dz}{dw} = \frac{P(w-c)^{\frac{1}{2}} + Q(w-1)^{\frac{1}{2}}}{w^{\frac{1}{2}}(w-1)^{\frac{1}{2}}(w-c)^{\frac{1}{2}}}, \tag{C 1}$$

where $c > 1$, P and Q are constants. This maps one half of the channel, as shown in figure 1(a), onto the upper half of the w -plane (Figure 1 b). This is mapped onto a uniform channel in the ζ -plane (figure 1 c) by the mapping

$$\frac{d\zeta}{dw} = \frac{S}{w^{\frac{1}{2}}(w-1)^{\frac{1}{2}}}. \tag{C 2}$$

In order to make both channels have width h , we set

$$\pi(P+Q) = h \tag{C 3}$$

and

$$\pi S = h. \tag{C 4}$$

Integrating (C 1) gives
$$z = 2iQ \cos^{-1}\left(\frac{1}{c^{\frac{1}{2}}}\right) \text{ at } C \tag{C 5}$$

and

$$z = 2P \cosh^{-1}(c^{\frac{1}{2}}) \text{ at } D. \tag{C 6}$$

If CD is to be a quadrant of a circle, with radius a , we obtain

$$a = 2P \cosh^{-1}(\sec \beta) = 2Q\beta \tag{C 7}$$

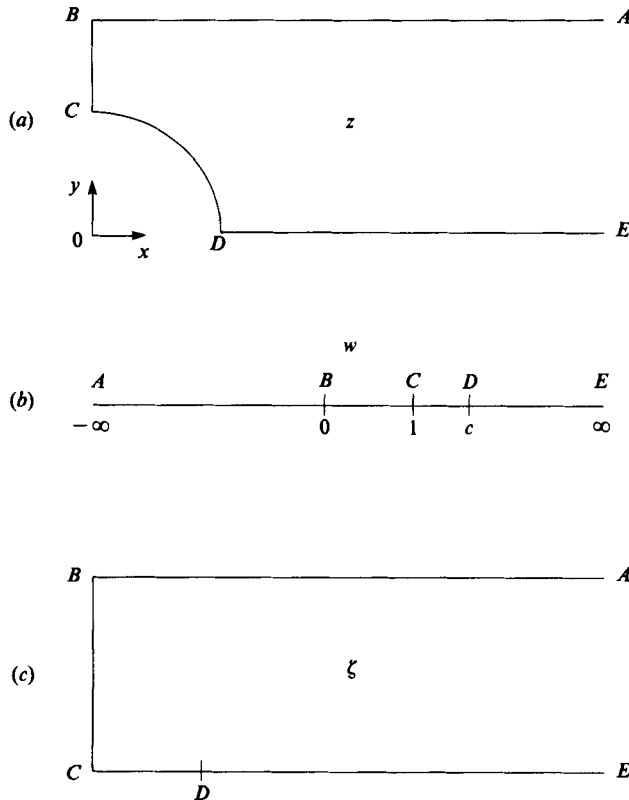


FIGURE 1. (a) Quadrant in a channel, in the $z = x + iy$ plane. (b) Mapping on w -plane, as given by (C 1). (c) Mapping on ζ -plane, as given by (C 2).

from (C 5) and (C 6), were we have defined an angle β by

$$\sec \beta = c^{\frac{1}{2}}. \tag{C 8}$$

Thus, given a/h , we can determine P/h , Q/h , and β from (C 3) and (C 7). Eliminating P and Q , we obtain a single equation, (73), for β . Richmond has determined β for various values of a/h . He has also shown that the actual curve CD departs from a circular arc by a maximum of 2% for $a/h = 0.5$ and 11% for $a/h = 0.75$.

In order to calculate the blockage coefficient l for uniform flow past a semicircular protrusion, we must examine the behaviour of

$$\phi(z) = \text{Re}(\zeta(z)) \quad \text{as } x \rightarrow +\infty. \tag{C 9}$$

From (C 1) and (C 2), we have

$$z = 2P \cosh^{-1}(w^{\frac{1}{2}}) + 2Q \cosh^{-1}(w^{\frac{1}{2}} \cos \beta) \tag{C 10}$$

and
$$\zeta = 2S \cosh^{-1}(w^{\frac{1}{2}}) \tag{C 11}$$

for $w > c$. For large w (i.e. near E), (C 10) gives

$$z = 2(P + Q) \cosh^{-1}(w^{\frac{1}{2}}) + 2Q \log(\cos \beta) + O(w^{-1}) \tag{C 12}$$

as $w \rightarrow \infty$. Hence, using (C 3), (C 4) and (C 11), we obtain

$$\zeta \sim z - 2Q \log(\cos \beta), \tag{C 13}$$

whence, comparing with (31), and using (C 7), we obtain finally

$$l = 2Q \log(\sec \beta) = \frac{\alpha}{\beta} \log(\sec \beta). \tag{C 14}$$

Appendix D. Asymmetric cylinders: a systematic approach

Another approach to our modified matched-asymptotic-expansion method has been provided by A. F. Messiter. The technique involves assumed power series for D_a and D_s in terms of kh .

D.1. *Acoustics*

The systematic approach, using powers of kh , shows how the Poisson equation arises. One advantage of the approach is that the cylinder need not be symmetric about $x = 0$. The amplitudes D_s and D_a in (39) depend on kh ; assume that

$$D_s = \sum_{n=1} \alpha_n (ikh)^n, \quad D_a = \sum_{n=1} \beta_n (ikh)^n, \tag{D 1}$$

where the coefficients α_n and β_n do not depend on kh . The leading terms α_1 and β_1 can be obtained as follows. The inner expansion of the outer solution is

$$\phi = e^{ikx} + (D_s + D_a \operatorname{sgn} x) e^{ik|x|} \tag{D 2}$$

$$\sim 1 + (ikh) \{ \bar{x} + (\alpha_1 + \beta_1 \operatorname{sgn} x) \} - (kh)^2 \{ \frac{1}{2} \bar{x}^2 + |\bar{x}| (\alpha_1 + \beta_1 \operatorname{sgn} x) + (\alpha_2 + \beta_2 \operatorname{sgn} x) \} \tag{D 3}$$

as $k|x| \rightarrow 0$, where $\bar{x} = x/h$ and $\bar{y} = y/h$ are the inner variables. In the inner region, suppose that

$$\phi = 1 + (ikh) \phi_1(\bar{x}, \bar{y}) - (kh)^2 \phi_2(\bar{x}, \bar{y}) - \dots, \tag{D 4}$$

whence (2) implies that $(\phi_1)_{\bar{x}\bar{x}} + (\phi_1)_{\bar{y}\bar{y}} = 0$ (D 5)

and $(\phi_2)_{\bar{x}\bar{x}} + (\phi_2)_{\bar{y}\bar{y}} = 1$, (D 6)

together with boundary conditions on the walls. In order to match with outer solutions we require that

$$\phi_1(\bar{x}, \bar{y}) \sim \bar{x} + (\alpha_1 + \beta_1 \operatorname{sgn} x) \tag{D 7}$$

and $\phi_2(\bar{x}, \bar{y}) \sim \frac{1}{2} \bar{x}^2 + |\bar{x}| (\alpha_1 + \beta_1 \operatorname{sgn} x) + (\alpha_2 + \beta_2 \operatorname{sgn} x)$ (D 8)

as $|\bar{x}| \rightarrow \infty$. A comparison of ϕ_1 with ψ_a shows that

$$\phi_1(\bar{x}, \bar{y}) = \psi_a(x, y) + \alpha_1, \tag{D 9}$$

whence (31), (D 4) give $\beta_1 = \frac{l}{h}$. (D 10)

To obtain α_1 , we must use properties of ϕ_2 . Thus, a conservation argument applied to ϕ_2 (as in §2.2.2) gives

$$\alpha_1 = -\frac{S}{2h^2} = M, \tag{D 11}$$

i.e. $\alpha_1 = \frac{m}{h}$. (D 12)

(For symmetric cylinders, we see that

$$\phi_2(\bar{x}, \bar{y}) = \psi_s(x, y) + \beta_1 \psi_a(x, y) + \alpha_2, \quad (D 13)$$

whence (52), (54), (60) and (D 8) give (D 12) and also

$$\beta_2 = (l/h)^2. \quad (D 14)$$

Substituting (D 10) and (D 12) into (D 1) gives

$$D_s \approx ikh\alpha_1 = ikm, \quad D_a \approx ikh\beta_1 = ikl, \quad (D 15)$$

in agreement with (46) and (62), to leading order in kh .

D.2. Water waves

A similar calculation succeeds for shallow-water waves. The inner expansion of the outer solution is

$$\phi = Y(y) \{e^{ikx} + (D_s + D_a \operatorname{sgn} x) e^{ik|x|}\} \quad (D 16)$$

$$\sim 1 + ikh\{\bar{x} + (\alpha_1 + \beta_1 \operatorname{sgn} x)\} - (kh)^2 \left\{ \frac{1}{2}(\bar{x}^2 - \bar{y}^2 + 2\bar{y}) + |\bar{x}|(\alpha_2 + \beta_1 \operatorname{sgn} x) + (\alpha_1 + \beta_2 \operatorname{sgn} x) \right\} \quad (D 17)$$

as $k|x| \rightarrow 0$, where D_s and D_a are given by (D 1). In the inner region, we assume that (D 4) holds. Now, ϕ_1 and ϕ_2 are both harmonic, but the free-surface condition, (93) and (91), gives

$$(\phi_1)_{\bar{y}} = 0, \quad (\phi_2)_{\bar{y}} = 1 \quad (D 18)$$

on $y = 0$. Similar arguments now show that

$$\alpha_1 = \frac{n}{h}, \quad \beta_1 = \frac{l}{h}. \quad (D 19)$$

(Indeed, we have (D 9) and, for symmetric cylinders, $\phi_2(\bar{x}, \bar{y}) = \hat{\psi}_s(x, y) + \beta_1 \psi_a(x, y) + \alpha_2$.) If we substitute (D 19) into (D 1), we obtain agreement with (107) and (122), to leading order in kh .

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